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Berezin transforms and Laplace–Beltrami operators on homogeneous Siegel domains

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Abstract: In this paper we show that on a homogeneous Siegel domain, commutativity of Berezin transforms with the Laplace–Beltrami operator is equivalent to the symmetry of the domain.

Keywords: Berezin transform, homogeneous Siegel domain, normal j-algebra, Laplace–Beltrami operator.

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1. Introduction

Berezin transforms on symmetric Siegel domains (or equivalently on bounded symmetric domains) are now fairly understood through their spectral decomposition due to [3, 22], and their role as a generating set of invariant operators shown by [6]. However, no works have been done concerning Berezin transforms on general Siegel domains. This paper begins the study of Berezin transforms on homogeneous Siegel domains without symmetry condition.

In the symmetric cases, comparison of Berezin transforms with invariant differential operators has worked well, because the algebra of invariant differential operators is commutative. But by [1, Theorem 8], this algebra is no longer commutative when the domain is not symmetric. Moreover the result of [5] concerning Berezin transforms on domains in $\mathbb{C}$ suggests that commutativity of Berezin transforms with the Laplace–Beltrami operator might prescribe the nature of the domain itself. We confirm these circumstances by showing that Berezin transforms on a homogeneous Siegel domain commute with the Laplace–Beltrami operator if and only if the domain is symmetric. The actual statement of our theorem (Theorem 4.10) is a little stronger than this and formulated by using admissible linear forms on the corresponding normal j-algebra.

In order to state our theorem concretely, let us fix the notation. Let D be an irreducible homogeneous Siegel domain and G the split solvable Lie group acting simply transitively on D. The Lie algebra $\mathfrak{g}$ of G has a structure of normal j-algebra ([18]). Thus there is an admissible linear form ω on $\mathfrak{g}$, so that we have a real inner product $\langle x \mid y\rangle_\omega := \langle [Jx, y], \omega\rangle$ on $\mathfrak{g}$, where J is the almost complex structure on $\mathfrak{g}$ attached to the normal j-algebra structure of $\mathfrak{g}$. The inner product $\langle \cdot \mid \cdot\rangle_\omega$ defines a left invariant Riemannian structure on G, through which

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we have the Laplace–Beltrami operator $\mathcal{L}_\omega$ on G. This type of Laplace–Beltrami operator is expressed in terms of elements of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ as Urakawa [23, Theorem 1] has shown for general connected Lie groups. The operator $\mathcal{L}_\omega$ corresponds to $-\Lambda + \Psi \in U(\mathfrak{g})$ as a left invariant differential operator, where Λ is the square-sum of orthonormal basis elements of $\mathfrak{g}$, and $\Psi \in \mathfrak{g}$ is the element for which $\langle x \mid \Psi \rangle_\omega = \operatorname{tr} \operatorname{ad} x$ holds for any $x \in \mathfrak{g}$. On the other hand, our Berezin transforms B_λ, when transferred to operators on $L^2(G)$, are convolution operators by functions a_λ from the right (see (3.12); here the constants c_λ there concern nothing). With these observations we proceed as follows:

(1) B_λ commutes with $\mathcal{L}_\omega$ if and only if the action of the element $\Lambda - \Psi$ on the function a_λ as a left invariant differential operator is the same as the action as a right invariant one (Proposition 4.1),

(2) B_λ commutes with $\mathcal{L}_\omega$ if and only if $(\Lambda - \Psi)\, a_\lambda(x) = (\Lambda - \Psi)\, a_\lambda(x^{-1})$ for any $x \in G$ (Proposition 4.2),

(3) $(\Lambda - \Psi)\, a_\lambda(x) = \lambda\, a_\lambda(x)\big(\lambda\, \|\mathcal{C}(x \cdot \mathsf{e})\|_\omega^2 - \langle \Psi, \alpha \rangle\big)$ for some $\alpha \in \mathfrak{g}^*$, where $\mathcal{C}$ is the Cayley transform introduced in [14], and $\mathsf{e} \in D$ is the reference point specified in (2.12) (Proposition 4.9).

Since $a_\lambda(x) = a_\lambda(x^{-1})$ for any $x \in G$, our commutativity problem is thus reduced to the validity of $\|\mathcal{C}(x \cdot \mathsf{e})\|_\omega = \|\mathcal{C}(x^{-1} \cdot \mathsf{e})\|_\omega$. This norm equality is studied in the previous paper [15]. Making use of [15, Theorem 5.3], we conclude in Theorem 4.10 that B_λ commutes with $\mathcal{L}_\omega$ if and only if D is symmetric and ω is proportional to the Koszul form β (see (4.10)) on the derived algebra $[\mathfrak{g}, \mathfrak{g}]$.

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2. Preliminaries

2.1. Normal j-algebras

Let $\mathfrak{g}$ be a split solvable Lie algebra, J a linear operator on $\mathfrak{g}$ with $J^2 = -I$ and ω a linear form on $\mathfrak{g}$. Then the triple $(\mathfrak{g}, J, \omega)$ is called a *normal j-algebra* if

$$[Jx, Jy] = [x, y] + J[Jx, y] + J[x, Jy] \quad \text{for all } x, y \in \mathfrak{g}, \tag{2.1}$$

$$\langle x \mid y \rangle_\omega := \langle [Jx, y], \omega \rangle \quad \text{defines a } J\text{-invariant inner product on } \mathfrak{g}. \tag{2.2}$$

Linear forms ω on $\mathfrak{g}$ which satisfy (2.2) are said to be *admissible*. In this subsection, we present some basic facts about normal j-algebras following [18] and [20] (see also [19]). Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra. Let $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ be the derived algebra of $\mathfrak{g}$, and $\mathfrak{a}$ the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{g}$ relative to the inner product $\langle \cdot \mid \cdot \rangle_\omega$. Then $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$. We know that $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ such that every operator in $\operatorname{ad}(\mathfrak{a})$ is semisimple on $\mathfrak{g}$. Thus we have a simultaneous eigenspace decomposition $\mathfrak{g} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{n}_\alpha$, where Δ is a finite subset of $\mathfrak{a}^*$ and

$$\mathfrak{n}_\alpha := \{x \in \mathfrak{n}\ ;\ [h, x] = \langle h, \alpha \rangle\, x \quad \text{for all } h \in \mathfrak{a}\}.$$

The dimension $r := \dim \mathfrak{a}$ is called the rank of the normal j-algebra. One can choose a basis $H_1, \ldots, H_r$ of $\mathfrak{a}$ such that if we set $E_j := -JH_j$, then $[H_j, E_k] = \delta_{jk} E_k$. Let $\alpha_1, \ldots, \alpha_r$ be the basis of $\mathfrak{a}^*$ dual to $H_1, \ldots, H_r$. Then elements of Δ, which we call the *roots* of $\mathfrak{g}$, are of the following form (not all possibilities need occur):

$$\begin{array}{llll} \frac{1}{2}(\alpha_m + \alpha_k), & 1 \leqslant k < m \leqslant r; & \frac{1}{2}(\alpha_m - \alpha_k), & 1 \leqslant k < m \leqslant r; \\ \frac{1}{2}\alpha_k, & 1 \leqslant k \leqslant r; & \alpha_k, & 1 \leqslant k \leqslant r. \end{array} \tag{2.3}$$

We remark that $\mathfrak{n}_{\alpha_k} = \mathbb{R}E_k$ and that if α, β are distinct roots, then $\mathfrak{n}_\alpha$ is orthogonal to $\mathfrak{n}_\beta$. Put

$$H := H_1 + \cdots + H_r, \qquad E := E_1 + \cdots + E_r. \tag{2.4}$$

Then we have the eigenspace decomposition $\mathfrak{g} = \mathfrak{g}(0) + \mathfrak{g}(1/2) + \mathfrak{g}(1)$, where

$$\mathfrak{g}(0) := \mathfrak{a} \oplus \sum_{m>k} \mathfrak{n}_{\frac{1}{2}(\alpha_m - \alpha_k)}, \qquad \mathfrak{g}(1/2) := \sum_{i=1}^{r} \mathfrak{n}_{\frac{1}{2}\alpha_i},$$

$$\mathfrak{g}(1) := \sum_{i=1}^{r} \mathfrak{n}_{\alpha_i} \oplus \sum_{m>k} \mathfrak{n}_{\frac{1}{2}(\alpha_m + \alpha_k)}.$$

Clearly, $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$ by understanding $\mathfrak{g}(i) = 0$ for $i > 1$. Moreover

$$J\mathfrak{n}_{\frac{1}{2}(\alpha_m - \alpha_k)} = \mathfrak{n}_{\frac{1}{2}(\alpha_m + \alpha_k)}, \quad m > k; \qquad J\mathfrak{n}_{\frac{1}{2}\alpha_i} = \mathfrak{n}_{\frac{1}{2}\alpha_i}, \quad 1 \leqslant i \leqslant r; \tag{2.5}$$

so that $J\mathfrak{g}(0) = \mathfrak{g}(1)$ and $J\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$. We remark here that the action of J on the elements of $\mathfrak{g}(0)$ is described as

$$JT = -[T, E], \quad T \in \mathfrak{g}(0); \qquad JT_{ji} = -[T_{ji}, E_i], \quad T_{ji} \in \mathfrak{n}_{\frac{1}{2}(\alpha_j - \alpha_i)}. \tag{2.6}$$

The following is the list of constants used frequently in this paper:

$$\begin{array}{ll} n_{mk} := \dim_{\mathbb{R}} \mathfrak{n}_{\frac{1}{2}(\alpha_m - \alpha_k)} = \dim_{\mathbb{R}} \mathfrak{n}_{\frac{1}{2}(\alpha_m + \alpha_k)}, & 1 \leqslant k < m \leqslant r; \\ p_j := \sum_{k>j} n_{kj}, \quad q_j := \sum_{i<j} n_{ji}, & 1 \leqslant j \leqslant r; \\ b_j := \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{n}_{\frac{1}{2}\alpha_j}, \quad d_j := 1 + \frac{1}{2}(p_j + q_j), & 1 \leqslant j \leqslant r; \\ \omega_k := \langle E_k, \omega \rangle = \|E_k\|_\omega^2 > 0, & 1 \leqslant k \leqslant r. \end{array} \tag{2.7}$$

2.2. Homogeneous Siegel domains

Let $(\mathfrak{g}, J, \omega)$ be a normal j-algebra, and $G = \exp \mathfrak{g}$ the connected and simply connected Lie group corresponding to $\mathfrak{g}$. Note that $\mathfrak{g}(0)$ is a Lie subalgebra of $\mathfrak{g}$. We denote by $G(0)$ the corresponding subgroup $\exp \mathfrak{g}(0)$ of G. By 2.1, we know that $G(0)$ acts on $V := \mathfrak{g}(1)$ by adjoint action. Recall $E \in V$ in (2.4) and let Ω be the $G(0)$-orbit through E. By [20, Theorem 4.15] Ω is a regular open convex cone in V, and $G(0)$ acts on Ω simply transitively. By (2.5) the subspace $\mathfrak{g}(1/2)$ is invariant under J, so that it is considered as a *complex* vector space by means of $-J$. We shall write this complex vector space by U. We put $W := V_{\mathbb{C}}$, the complexification of V. The conjugation of W relative to the real form V is written as $w \mapsto w^*$. The real bilinear map Q defined by

$$Q(u,u') := \tfrac{1}{2}\left([Ju,u'] - i[u,u']\right), \qquad u,u' \in \mathfrak{g}(1/2) \tag{2.8}$$

turns out to be a complex sesqui-linear (complex linear in the first variable and antilinear in the second) Hermitian map $U \times U \to W$ which is Ω-positive. This means that

$$Q(u',u) = Q(u,u')^*, \quad u,u' \in U;$$

$$Q(u,u) \in \overline{\Omega} \setminus \{0\}, \qquad \text{for all } u \in U \setminus \{0\}.$$

With these data we define the *Siegel domain* $D = D(\Omega, Q)$ by

$$D := \{(u,w) \in U \times W;\ w + w^* - Q(u,u) \in \Omega\}. \tag{2.9}$$

Every homogeneous Siegel domain arises in this way. In this paper we assume that D is irreducible.

Consider $\mathfrak{n}_D := \mathfrak{g}(1) + \mathfrak{g}(1/2)$. It is at most 2-step nilpotent by 2.1. Let $N_D = \exp \mathfrak{n}_D$ be the corresponding connected and simply connected nilpotent Lie group contained in G. Writing the elements of N_D by $n(a,b)$ ($a \in \mathfrak{g}(1)$, $b \in \mathfrak{g}(1/2)$), we see by the Campbell–Hausdorff formula that the group operation is described as (with Q as in (2.8))

$$n(a,b)\, n(a',b') = n(a + a' - \operatorname{Im} Q(b,b'),\, b + b'). \tag{2.10}$$

The group N_D acts on D by

$$\begin{aligned} &n(a,b)\cdot(u,w) = (u+b,\ w + ia + \tfrac{1}{2}\, Q(b,b) + Q(u,b)); \\ &\qquad (u,w) \in D. \end{aligned} \tag{2.11}$$

On the other hand, the adjoint action of $G(0)$ on $\mathfrak{g}(1/2)$ commutes with J. In other words, $G(0)$ acts on U complex-linearly. Moreover the adjoint action of $G(0)$ on $V = \mathfrak{g}(1)$ extends complex-linearly to W, so that $G(0)$ acts on D complex-linearly. Hence $G = N_D \rtimes G(0)$ acts on D simply transitively. To see this more directly, we consider the following reference point:

$$\mathsf{e} := (0, E) \in D. \tag{2.12}$$

Then given $z = (u,w) \in D$, we can find a unique $h \in G(0)$ satisfying $hE = \operatorname{Re} w - \frac{1}{2} Q(u,u)$. Taking $n = n(\operatorname{Im} w, u) \in N_D$, we get $z = nh \cdot \mathsf{e}$ by (2.11).

For every $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ let $\chi_{\mathbf{s}}$ be the one-dimensional representation of $A := \exp \mathfrak{a}$ defined by

$$\chi_{\mathbf{s}}\Bigl(\exp \textstyle\sum_k t_k H_k\Bigr) = \exp\Bigl(\displaystyle\sum_k s_k t_k\Bigr), \qquad t_1, \ldots, t_r \in \mathbb{R}. \tag{2.13}$$

We put

$$\mathfrak{n}_0 := \sum_{m>k} \mathfrak{n}_{\frac{1}{2}(\alpha_m - \alpha_k)}. \tag{2.14}$$

Clearly $\mathfrak{n}_0$ is a nilpotent Lie subalgebra of $\mathfrak{g}(0)$, and we have $\mathfrak{n} = \mathfrak{n}_0 + \mathfrak{n}_D$. Let $N_0 := \exp \mathfrak{n}_0$ and $N := \exp \mathfrak{n}$. It is also clear that $G = N \rtimes A$ and $G(0) = N_0 \rtimes A$. We extend $\chi_{\mathbf{s}}$ to a one-dimensional representation of G by defining $\chi_{\mathbf{s}}(n) = 1$ for $n \in N$. Let us define functions $\Delta_{\mathbf{s}}$ ($\mathbf{s} \in \mathbb{C}^r$) on Ω by

$$\Delta_{\mathbf{s}}(hE) = \chi_{\mathbf{s}}(h), \qquad h \in G(0). \tag{2.15}$$

Evidently it holds that

$$\Delta_{\mathbf{s}}(hx) = \chi_{\mathbf{s}}(h)\Delta_{\mathbf{s}}(x), \qquad h \in G(0),\ x \in \Omega. \tag{2.16}$$

In particular, putting $h = \exp tH \in A$ with $t = \log\lambda$ ($\lambda > 0$), we see that $\Delta_{\mathbf{s}}(\lambda x) = \lambda^{|\mathbf{s}|}\Delta_{\mathbf{s}}(x)$, where $|\mathbf{s}| := s_1 + \cdots + s_r$ for $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$. Furthermore, we know that $\Delta_{\mathbf{s}}$ extends to a holomorphic function on the tube domain $\Omega + iV$ (cf. for example [10, Corollary 2.5]).

For $h \in G(0)$, let $\mathrm{Ad}_{\mathfrak{g}(1)}(h) := (\mathrm{Ad}\,h)|_{\mathfrak{g}(1)}$. Moreover let $\mathrm{Ad}_U(h)$ stand for the *complex* linear operator on U defined by the adjoint action of $h \in G(0)$ on $\mathfrak{g}(1/2)$, and $\det \mathrm{Ad}_U(h)$ its determinant as a complex linear operator. Then, with $\mathbf{d} := (d_1, \ldots, d_r)$ and $\mathbf{b} := (b_1, \ldots, b_r)$, we have for $h \in G(0)$

$$\det \mathrm{Ad}_{\mathfrak{g}(1)}(h) = \chi_{\mathbf{d}}(h), \qquad |\det \mathrm{Ad}_U(h)|^2 = \chi_{\mathbf{b}}(h). \tag{2.17}$$

2.3. Bergman kernel

By [9, §5] or [21, §II.6], it is known that D has a Bergman kernel κ. If $\mathrm{Hol}\,(D)$ denotes the Lie group of the holomorphic automorphisms of D, then κ satisfies

$$\begin{aligned} \kappa(z_1, z_2) &= \kappa(g \cdot z_1, g \cdot z_2) \det g'(z_1)\, \overline{\det g'(z_2)}, \\ &\quad g \in \mathrm{Hol}\,(D),\ \ z_1, z_2 \in D, \end{aligned} \tag{2.18}$$

where $g'(z)$ is the complex Jacobian map of g at $z \in D$. By the discussion in 2.2 of the simple transitive action of G on D, (2.17) and (2.18) give us

$$\kappa(z_1, z_2) = C \cdot \Delta_{-2\mathbf{d}-\mathbf{b}}(w_1 + w_2^* - Q(u_1, u_2)), \qquad z_j = (u_j, w_j) \in D \tag{2.19}$$

with $C = \kappa(\mathrm{e}, \mathrm{e})\, \Delta_{2\mathbf{d}+\mathbf{b}}(2E) > 0$. We put $\eta := \Delta_{-2\mathbf{d}-\mathbf{b}}$ in what follows for simplicity.

2.4. Pseudoinverse map

Let D_v be the directional derivative in the direction $v \in V$ given by

$$D_v f(x) = \frac{d}{dt} f(x + tv)\bigg|_{t=0}.$$

For every $x \in \Omega$ we define $\mathcal{I}(x) \in V^*$ by

$$\langle v, \mathcal{I}(x)\rangle = -D_v \log \eta(x), \qquad v \in V. \tag{2.20}$$

The map $\mathcal{I}$ is called the *pseudoinverse map*. By [4, §2], $\mathcal{I}$ gives a diffeomorphism of Ω onto the dual cone Ω^* in V^*, where

$$\Omega^* := \{\, \xi \in V^*;\ \langle x, \xi\rangle > 0 \quad \text{for all } x \in \overline{\Omega} \setminus \{0\}\}.$$

Now the group $G(0)$ acts on V^* by the coadjoint action: $h \cdot \xi = \xi \circ h^{-1}$, where $h \in G(0)$ and $\xi \in V^*$. It is easy to show by using (2.16) that $\mathcal{I}$ is $G(0)$-equivariant

$$\mathcal{I}(hx) = h \cdot \mathcal{I}(x), \qquad h \in G(0),\ x \in \Omega. \tag{2.21}$$

In particular, $\mathcal{I}(\lambda x) = \lambda^{-1}\mathcal{I}(x)$ for all $\lambda > 0$, and $G(0)$ acts on Ω^* simply transitively. We know that $\mathcal{I}$ extends to a birational map $W \to W^*$. Moreover $\mathcal{I}$ maps the tube domain $\Omega + iV$ biholomorphically onto its image $\mathcal{I}(\Omega + iV)$ (see [14, §2]).

2.5. Cayley transform

First we define $E_1^*, \dots, E_r^* \in V^*$ by

$$\left\langle \sum_{j=1}^{r} x_j E_j + \sum_{m>k} X_{mk},\ E_i^* \right\rangle = x_i, \qquad x_j \in \mathbb{R},\ X_{mk} \in \mathfrak{n}_{\frac{1}{2}(\alpha_m+\alpha_k)}.$$

Elements of V^* are canonically considered as elements of W^*, the space of complex linear forms on W. Then for every $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ we set

$$E_{\mathbf{s}}^* := s_1 E_1^* + \cdots + s_r E_r^* \in W^*.$$

Now we define

$$C(w) := E_{2\mathbf{d}+\mathbf{b}}^* - 2\,\mathcal{I}(w + E) \in W^*, \qquad w \in W. \tag{2.22}$$

It is evident that C is a rational mapping $W \to W^*$ which is holomorphic on $\Omega + iV$. Let $U^\dagger$ denote the space of all antilinear forms on U. We set for $z = (u, w) \in U \times W$

$$\mathcal{C}(z) := (2\,\mathcal{I}(w + E) \circ Q(u, \cdot), C(w)) \in U^\dagger \times W^*. \tag{2.23}$$

Clearly $\mathcal{C}$ is a rational map $U \times W \to U^\dagger \times W^*$. It should be noted that if $z = (u, w) \in D$, then we have $w \in \Omega + iV$, so that $\mathcal{C}(z)$ is holomorphic on D. We shall call $\mathcal{C}$ a *Cayley transform*, which is introduced in [14] as a slight modification of Penney's [16]. We know by [14, §3] that $\mathcal{C}$ is a birational map which sends D biholomorphically onto a bounded domain $\mathcal{C}(D)$.

3. Berezin kernels and Laplace–Beltrami operators

3.1. Weighted Bergman spaces

First of all we introduce an inner product $\langle \cdot \mid \cdot \rangle_\eta$ on V by

$$\langle v_1 \mid v_2 \rangle_\eta := D_{v_1} D_{v_2} \log \eta(E), \qquad v_1, v_2 \in V. \tag{3.1}$$

We know by [14, Lemma 2.2] that

$$\langle v_1 \mid v_2 \rangle_\eta = \langle [J v_1, v_2], E_{2\mathbf{d}+\mathbf{b}}^* \rangle. \tag{3.2}$$

We extend $\langle \cdot \,|\, \cdot \rangle_\eta$ to a complex bilinear form on $W \times W$, which we still denote by the same symbol. Then we have a Hermitian inner product $(w_1 \,|\, w_2)_\eta := \langle w_1 \,|\, w_2^* \rangle_\eta$ on W. On the other hand, it is easy to see that U has a Hermitian inner product $(\cdot \,|\, \cdot)_\eta$ given by

$$(u_1 \,|\, u_2)_\eta := \langle Q(u_1, u_2) \,|\, E \rangle_\eta, \qquad u_1, u_2 \in U. \tag{3.3}$$

We denote by $dm(w)$ and $dm(u)$ the Euclidean measure normalized by these inner products on W and on U respectively. By (2.17), the measure $d\mu$ defined by

$$d\mu(u, w) := \eta(w + w^* - Q(u, u))\, dm(u)\, dm(w) \tag{3.4}$$

is a G-invariant measure on D.

Let us set for $\lambda \in \mathbb{R}$

$$d\mu_\lambda(u, w) := c_\lambda \cdot \eta(w + w^* - Q(u, u))^{-\lambda+1}\, dm(u)\, dm(w), \tag{3.5}$$

where $c_\lambda > 0$ is determined shortly. The *weighted Bergman space* $H^2_\lambda(D)$ is the Hilbert space of holomorphic functions on D which are square integrable relative to $d\mu_\lambda$. We know by [20, Theorem 4.26] or [10, Theorem 4.8] that $H^2_\lambda(D) \neq \{0\}$ if and only if

$$\lambda > \lambda_0 := \max_{1 \leqslant k \leqslant r} \frac{b_k + d_k + \frac{1}{2} p_k}{b_k + 2d_k}.$$

In view of (2.7), we have $0 < \lambda_0 < 1$. If $\lambda > \lambda_0$, the Hilbert space $H^2_\lambda(D)$ has a reproducing kernel κ_λ (cf. [10, Proposition 4.6]). We choose the constant c_λ in such a way that

$$\kappa_\lambda(z_1, z_2) = \eta(w_1 + w_2^* - Q(u_1, u_2))^\lambda, \qquad z_j := (u_j, w_j) \in D. \tag{3.6}$$

Explicit expression of c_λ is not necessary in this paper. The *Berezin kernel* $A_\lambda(z_1, z_2)$ is now defined to be

$$A_\lambda(z_1, z_2) := \frac{|\kappa_\lambda(z_1, z_2)|^2}{\kappa_\lambda(z_1, z_1)\, \kappa_\lambda(z_2, z_2)}, \qquad z_1, z_2 \in D. \tag{3.7}$$

By (2.18) and (2.19) A_λ is G-invariant

$$A_\lambda(g \cdot z_1,\ g \cdot z_2) = A_\lambda(z_1, z_2), \qquad g \in G. \tag{3.8}$$

3.2. Berezin transforms

Let us fix $\lambda > \lambda_0$ from now on. Consider the measure $d\mu_0(z) := \kappa_\lambda(z, z)\, d\mu_\lambda(z)$. By (3.4), (3.5) and (3.6), we see that $d\mu_0 = c_\lambda\, d\mu$. The Berezin transform B^D_λ is, by definition, the following integral operator on $L^2(D, d\mu_0)$

$$B^D_\lambda f(z) := \int_D A_\lambda(z, w)\, f(w)\, d\mu_0(w), \qquad f \in L^2(D, d\mu_0). \tag{3.9}$$

By [2] we know that B^D_λ is a bounded positive selfadjoint operator with $\|B^D_\lambda\| \leqslant 1$. Since G is diffeomorphic to D by the orbit map $g \mapsto g \cdot \mathrm{e}$ and since $d\mu_0$ is a G-invariant measure on D, we shall transfer the operator B_λ^D to $L^2(G)$ for left Haar measure. First let us normalize the left Haar measure dx on G so that

$$\int_G f(x\cdot \mathrm{e})\,dx = \int_D f(z)\,d\mu(z) = \frac{1}{c_\lambda}\int_D f(z)\,d\mu_0(z). \tag{3.10}$$

Then, $I_\lambda f(x) := c_\lambda^{\frac{1}{2}} f(x\cdot \mathrm{e})$ $(x \in G)$ gives a unitary isomorphism of $L^2(D, d\mu_0)$ onto $L^2(G)$. We set $B_\lambda := I_\lambda B_\lambda^D I_\lambda^{-1}$. Putting

$$a_\lambda(x) := A_\lambda(x\cdot \mathrm{e}, \mathrm{e}), \qquad x \in G, \tag{3.11}$$

we see easily that B_λ is the integral operator given by

$$B_\lambda f(x) = c_\lambda \int_G a_\lambda(y^{-1}x)\; f(y)\,dy = c_\lambda\, f * a_\lambda(x), \qquad f \in L^2(G), \tag{3.12}$$

where we have used the standard definition of convolution $*$ arising from the left translation (see, e.g., [8, p. 55]). We note here the following simple fact.

Lemma 3.1. *For $\lambda > \lambda_0$ one has $a_\lambda \in L^1(G)$.*

Proof. In view of (3.7) and (3.10), the reproducing property of κ_λ gives

$$\int_G a_\lambda(x)\,dx = \frac{c_\lambda^{-1}}{\kappa_\lambda(\mathrm{e},\mathrm{e})}\int_D |\kappa_\lambda(z,\mathrm{e})|^2\,d\mu_\lambda(z) = c_\lambda^{-1} < \infty. \tag{3.13}$$

Since $a_\lambda \geqslant 0$, this implies that a_λ is integrable over G. $\square$

3.3. Laplace–Beltrami operators

If $X \in \mathfrak{g}$, we set

$$Xf(x) := \frac{d}{dt} f(\exp(-tX)\,x)\bigg|_{t=0}, \qquad \widetilde{X}f(x) := \frac{d}{dt} f(x\,(\exp tX))\bigg|_{t=0},$$

where $f \in C^\infty(G)$ and $x \in G$. These two actions of $\mathfrak{g}$ on $C^\infty(G)$ are extended to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ by homomorphisms. We have the inner product $\langle\cdot\,|\,\cdot\rangle_\omega$ on $\mathfrak{g}$ by the definition (2.2). This inner product induces a left invariant Riemannian metric on G, relative to which the Laplace–Beltrami operator $\mathcal{L}_\omega$ is defined. Though the following lemma holds for any connected Lie group, we write it down here in our situation. See [23, Theorem 1] for a proof (see also [17, Proposition 2.2]).

Lemma 3.2. *Take $\Psi \in \mathfrak{g}$ such that $\langle X\,|\,\Psi\rangle_\omega = \operatorname{tr}\operatorname{ad}(X)$ holds for all $X \in \mathfrak{g}$. Then $\mathcal{L}_\omega = -\widetilde{\Lambda} + \widetilde{\Psi}$, where $\Lambda := X_1^2 + \cdots + X_{2N}^2 \in U(\mathfrak{g})$, $2N := \dim_{\mathbb{R}} \mathfrak{g}$, with an orthonormal basis $\{X_j\}_{j=1}^{2N}$ of $\mathfrak{g}$ relative to $\langle\cdot\,|\,\cdot\rangle_\omega$. Note that Λ remains the same under any change of orthonormal basis of $\mathfrak{g}$.*

Lemma 3.3. *With the constants in* (2.7), the element Ψ in Lemma 3.2 is given by

$$\Psi = \sum_{j=1}^{r} \omega_j^{-1}(q_j + b_j + 1)\,H_j \in \mathfrak{a}.$$

Proof. If $X \in \mathfrak{n}$, then ad X is a nilpotent operator on $\mathfrak{g}$. This implies

$$\langle \Psi \mid X \rangle_\omega = \operatorname{tr} \operatorname{ad} X = 0,$$

so that $\Psi \in \mathfrak{n}^\perp = \mathfrak{a}$. Suppose now that $H \in \mathfrak{a}$. Then with n_{ji} as in (2.7) we have

$$\begin{aligned} \operatorname{tr} \operatorname{ad} H &= \sum_{j>i} n_{ji} \cdot \tfrac{1}{2} \langle H, \alpha_j - \alpha_i \rangle + 2 \sum_j b_j \cdot \tfrac{1}{2} \langle H, \alpha_j \rangle \\ &\quad + \sum_{j>i} n_{ji} \cdot \tfrac{1}{2} \langle H, \alpha_j + \alpha_i \rangle + \sum_j \langle H, \alpha_j \rangle \\ &= \sum_j (q_j + b_j + 1) \langle H, \alpha_j \rangle. \end{aligned}$$

Since $\|H_j\|_\omega = \|E_j\|_\omega = \omega_j^{\frac{1}{2}}$ by (2.7), we have $\langle H, \alpha_j \rangle = \omega_j^{-1} \langle H \mid H_j \rangle_\omega$, which together with the above computation yields

$$\langle H \mid \Psi \rangle_\omega = \operatorname{tr} \operatorname{ad} H = \sum\nolimits_j \omega_j^{-1} (q_j + b_j + 1) \langle H \mid H_j \rangle_\omega.$$

Hence the proof is completed. □

4. Commutativity conditions

Recall that we are working with the left Haar measure dx on G. Let δ be the modular function (Haar modulus) of G. We know (see, e.g., [8, 2.4]) that $\delta(x) = \det \operatorname{Ad}(x^{-1})$ and that

$$d(xy) = \delta(y)\, dx. \tag{4.1}$$

Fix $\lambda > \lambda_0$ and consider the Berezin transform B_λ on $L^2(G)$ introduced in 3.2. Let $\mathcal{L}_\omega$ be the Laplace–Beltrami operator on G defined in 3.3. Let Λ be as in Lemma 3.2.

Proposition 4.1. *B_λ commutes with $\mathcal{L}_\omega$ if and only if*

$$(-\widetilde{\Lambda} + \widetilde{\Psi})\, a_\lambda = (-\Lambda + \Psi)\, a_\lambda.$$

Proof. Let $f \in C_c^\infty(G)$. Since $\mathcal{L}_\omega$ commutes with left translations, Lemma 3.2 and (3.12) give

$$\mathcal{L}_\omega B_\lambda f(x) = c_\lambda \int_G f(g)((-\widetilde{\Lambda} + \widetilde{\Psi})\, a_\lambda)(g^{-1}x)\, dg. \tag{4.2}$$

On the other hand, (3.12) and (4.1) yield that if $R_y f(x) := f(xy)$ $(x, y \in G)$, then

$$B_\lambda R_y f(x) = c_\lambda \cdot (\det \operatorname{Ad} y) \int_G f(g)\, a_\lambda(yg^{-1}x)\, dg.$$

Hence if $X \in \mathfrak{g}$, we get by the definition of Ψ

$$B_\lambda \widetilde{X} f(x) = \langle X \mid \Psi \rangle_\omega \cdot B_\lambda f(x) - c_\lambda \int_G f(g)(X a_\lambda)(g^{-1}x)\, dg, \tag{4.3}$$

so that we have

$$B_\lambda \widetilde{X}^2 f(x) = \langle X \mid \Psi\rangle_\omega \cdot B_\lambda \widetilde{X} f(x) - c_\lambda \int_G \widetilde{X} f(g)(Xa_\lambda)(g^{-1}x)\, dg.$$

Here the integral of the second term on the right-hand side can be rewritten as

$$\frac{d}{dt}\left[e^{t\langle X \mid \Psi\rangle_\omega} \int_G f(g)(Xa_\lambda)(\exp tX \cdot g^{-1}x)\, dg\right]\Big|_{t=0}.$$

Therefore we get by (4.3)

$$\begin{aligned} B_\lambda \widetilde{X}^2 f(x) = {} & \langle X \mid \Psi\rangle_\omega^2 \cdot B_\lambda f(x) - 2c_\lambda \langle X \mid \Psi\rangle_\omega \int_G f(g)(Xa_\lambda)(g^{-1}x)\, dg \\ & + c_\lambda \int_G f(g)(X^2 a_\lambda)(g^{-1}x)\, dg. \end{aligned} \tag{4.4}$$

Taking an orthonormal basis $\{X_j\}_{j=1}^{2N}$ of $\mathfrak{g}$ and summing up the formula (4.4) over $X = X_1, \ldots, X_{2N}$, we obtain

$$\begin{aligned} B_\lambda \widetilde{\Lambda} f(x) = {} & \|\Psi\|_\omega^2 \cdot B_\lambda f(x) - 2c_\lambda \int_G f(g)(\Psi a_\lambda)(g^{-1}x)\, dg \\ & + c_\lambda \int_G f(g)(\Lambda a_\lambda)(g^{-1}x)\, dg. \end{aligned}$$

This together with Lemma 3.2 and the formula (4.3) for $X = \Psi$ leads us to

$$B_\lambda \mathcal{L}_\omega f(x) = B_\lambda(-\widetilde{\Lambda} + \widetilde{\Psi})\, f(x) = c_\lambda \int_G f(g)((-\Lambda + \Psi)\, a_\lambda)(g^{-1}x)\, dg.$$

Comparing this with (4.2) and noting that $f \in C_c^\infty(G)$ is arbitrary, we get the proposition. $\square$

We next make a general observation. Given a smooth function f on G, we define $f^\vee(x) := f(x^{-1})$ $(x \in G)$. Then by a simple calculation we have $\widetilde{X}(f^\vee) = (Xf)^\vee$ for any $X \in \mathfrak{g}$, and thus for any $X \in U(\mathfrak{g})$. In particular, if $f^\vee = f$, then $\widetilde{X} f = (Xf)^\vee$ for all $X \in U(\mathfrak{g})$.

Now the function a_λ satisfies $(a_\lambda)^\vee = a_\lambda$ owing to (3.7), (3.8) and (3.11). The above observation then yields

$$(\widetilde{\Lambda} - \widetilde{\Psi})\, a_\lambda(x) = (\Lambda - \Psi)\, a_\lambda(x^{-1}).$$

Therefore we obtain the following proposition.

Proposition 4.2. *B_λ commutes with $\mathcal{L}_\omega$ if and only if*

$$(\Lambda - \Psi)\, a_\lambda(x^{-1}) = (\Lambda - \Psi)\, a_\lambda(x) \qquad \textit{for any } x \in G.$$

We now compute $(\Lambda - \Psi)\, a_\lambda$. Put $\mathbf{c} := 2\mathbf{d} + \mathbf{b}$ for simplicity. Since $\eta = \Delta_{-\mathbf{c}}$ is a homogenous function with degree $-|\mathbf{c}|$, the formulas (2.11) and (3.6) give (see the discussion just after (2.12)) for $n \in N_D$ and $h \in G(0)$

$$\kappa_\lambda(nh \cdot \mathsf{e},\, nh \cdot \mathsf{e}) = 2^{-|\mathbf{c}|\lambda}\, \eta(hE)^\lambda, \qquad \kappa_\lambda(nh \cdot \mathsf{e},\, \mathsf{e}) = \eta(\pi_W(nh \cdot \mathsf{e}) + E)^\lambda,$$

where π_W stands for the projection $U \times W \to W$. Therefore we have

$$a_\lambda(nh) = 4^{|\mathbf{c}|\lambda}\chi_{\mathbf{c}}(h)^\lambda \left|\eta(\pi_W(nh\cdot \mathsf{e}) + E)\right|^{2\lambda}, \qquad n \in N_D,\ h \in G(0). \tag{4.5}$$

We record here the following simple formula frequently used in the sequel.

Lemma 4.3. *For $w \in W$ one has*

$$D_w\eta^\lambda(w_0) = -\lambda\,\langle w, \mathfrak{I}(w_0)\rangle \cdot \eta^\lambda(w_0), \qquad w_0 \in \Omega + iV,$$

where $\mathfrak{I}$ is the pseudoinverse map introduced in 2.4.

Proof. Observe that the definition (2.20) of $\mathfrak{I}$ gives $D_w\eta = -\langle w, \mathfrak{I}(\cdot)\rangle \cdot \eta$. $\square$

In what follows, we put $w_0 := \pi_W(n(v_0, u_0)\,h_0 \cdot \mathsf{e})$ for $v_0 \in V$, $u_0 \in U$ and $h_0 \in G(0)$. We have $w_0 = h_0E + iv_0 + \frac{1}{2}Q(u_0, u_0)$ by (2.11). We write n_0 instead of $n(v_0, u_0)$ for brevity.

Lemma 4.4. *If $Y \in \mathfrak{g}(1)$, then*

$$Y^2a_\lambda(n_0h_0) = \lambda a_\lambda(n_0h_0)\big[4\lambda\big(\mathrm{Im}\,\langle Y,\ \mathfrak{I}(w_0 + E)\rangle\big)^2 + 2\,\mathrm{Re}\,\langle Y,\ (D_Y\mathfrak{I})(w_0 + E)\rangle\big].$$

Proof. Since $(\exp - tY)\,n_0 = n(v_0 - tY, u_0)$, it holds that $\pi_W\big((\exp - tY)\,n_0h_0\cdot\mathsf{e}\big) = w_0 - itY$. Hence

$$\begin{aligned} Ya_\lambda(n_0h_0) =& 4^{|\mathbf{c}|\lambda}\chi_{\mathbf{c}}(h_0)^\lambda\,\frac{d}{dt}\left[\eta(w_0 + E - itY)^\lambda\,\eta(w_0^* + E + itY)^\lambda\right]\Big|_{t=0} \\ =& 4^{|\mathbf{c}|\lambda}\,i\cdot\chi_{\mathbf{c}}(h_0)^\lambda\big[-(D_Y\eta^\lambda)(w_0 + E)\eta^\lambda(w_0^* + E) \\ &\qquad + \eta^\lambda(w_0 + E)(D_Y\eta^\lambda)(w_0^* + E)\big], \end{aligned}$$

because holomorphy of η^λ implies that $W \ni w \mapsto D_w\eta^\lambda$ is complex-linear. Applying Lemma 4.3, we get

$$Ya_\lambda(n_0h_0) = i\lambda\cdot a_\lambda(n_0h_0)\big(\langle Y,\ \mathfrak{I}(w_0 + E)\rangle - \langle Y,\ \mathfrak{I}(w_0^* + E)\rangle\big).$$

Repeating this calculation and noting that $\mathfrak{I}$ is holomorphic on $\Omega + iV$, we get

$$\begin{aligned} Y^2a_\lambda(n_0h_0) =& \frac{d}{dt}\,(Ya_\lambda)((\exp - tY)\,n_0h_0)\Big|_{t=0} \\ =& \lambda a_\lambda(n_0h_0)\big[-\lambda\big(\langle Y,\ \mathfrak{I}(w_0 + E)\rangle - \langle Y,\ \mathfrak{I}(w_0^* + E)\rangle\big)^2 \\ &\qquad + \langle Y,\ (D_Y\mathfrak{I})(w_0 + E) + (D_Y\mathfrak{I})(w_0^* + E)\rangle\big], \end{aligned}$$

from which the lemma follows immediately. $\square$

Lemma 4.5. *If $T \in \mathfrak{g}(0)$, then putting $X := JT \in \mathfrak{g}(1)$, one has*

$$\begin{aligned} T^2a_\lambda(n_0h_0) =& \lambda a_\lambda(n_0h_0)\big[\lambda\big(-\langle X, E_{\mathbf{c}}^*\rangle + 2\,\mathrm{Re}\,\langle X,\ \mathfrak{I}(w_0 + E)\rangle\big)^2 \\ & -2\,\mathrm{Re}\,\langle X,\ (D_X\mathfrak{I})(w_0 + E)\rangle + 2\,\mathrm{Re}\,\langle [T, JT],\ \mathfrak{I}(w_0 + E)\rangle\big]. \end{aligned}$$

Before giving a proof, let us fix the notation. Let $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{R}^r$ and we define a linear form $\alpha_{\mathbf{s}}$ on $\mathfrak{a}$ by

$$\alpha_{\mathbf{s}}\Big(\sum_k t_k H_k\Big) = \sum_k s_k t_k, \qquad t_1, \dots, t_r \in \mathbb{R}. \tag{4.6}$$

We extend $\alpha_{\mathbf{s}}$ to an element of $\mathfrak{g}(0)^*$ by setting $\alpha_{\mathbf{s}}(\mathfrak{g}(0) \cap \mathfrak{a}^{\perp}) = 0$, so that we have $\chi_{\mathbf{s}}(\exp T) = \exp \alpha_{\mathbf{s}}(T)$ for any $T \in \mathfrak{g}(0)$. Since $JH_k = -E_k$ for any k by definition, we also have

$$\langle T, \alpha_{\mathbf{s}} \rangle = -\langle JT, E_{\mathbf{s}}^* \rangle, \qquad T \in \mathfrak{g}(0). \tag{4.7}$$

We now prove Lemma 4.5.

Proof. In this proof, we write e^T and $e^T z$ $(z \in U + W)$ instead of $\exp T$ and $\mathrm{Ad}(\exp T) z$ respectively. Since $e^{-tT} \cdot n_0 h_0 = n(e^{-tT} v_0, e^{-tT} u_0) e^{-tT} h_0$, we have

$$\begin{aligned} a_\lambda(e^{-tT} n_0 h_0) &= 4^{|\mathbf{c}|\lambda} \chi_{\mathbf{c}}(e^{-tT} h_0)^\lambda\, \eta(e^{-tT} w_0 + E)^\lambda\, \eta(e^{-tT} w_0^* + E)^\lambda \\ &= 4^{|\mathbf{c}|\lambda} \chi_{\mathbf{c}}(e^{tT} h_0)^\lambda\, \eta(w_0 + e^{tT} E)^\lambda\, \eta(w_0^* + e^{tT} E)^\lambda, \end{aligned}$$

where we used (2.16) for the second equality. Differentiation at $t = 0$ together with (2.6), (4.7) and Lemma 4.3 yields

$$T a_\lambda(n_0 h_0) = \lambda a_\lambda(n_0 h_0)\big[-\langle X, E_{\mathbf{c}}^* \rangle + 2\,\mathrm{Re}\,\langle X,\ \mathcal{I}(w_0 + E)\rangle\big]. \tag{4.8}$$

Since $\langle X,\ \mathcal{I}(e^{-tT} w_0 + E)\rangle = \langle e^{tT} X,\ \mathcal{I}(w_0 + e^{tT} E)\rangle$ by (2.21), it holds that

$$\frac{d}{dt}\langle X,\ \mathcal{I}(e^{-tT} w_0 + E)\rangle\Big|_{t=0} = \langle [T, X],\ \mathcal{I}(w_0 + E)\rangle - \langle X,\ (D_X \mathcal{I})(w_0 + E)\rangle,$$

where we used again (2.6). Therefore differentiating $T a_\lambda(e^{-tT} n_0 h_0)$ at $t = 0$ by using (4.8), we get the lemma easily. □

We set $c_j := 2d_j + b_j$ $(j = 1, 2, \dots, r)$ for brevity.

Proposition 4.6. (1) *For* $T \in \mathfrak{n}_{\frac{1}{2}(\alpha_k - \alpha_i)}$ $(k > i)$ *with* $\|T\|_\omega = 1$, *one has*

$$\begin{aligned} (T^2 + (JT)^2)\, a_\lambda(n_0 h_0) = 2\lambda a_\lambda(n_0 h_0)\big[&2\lambda\, |\langle JT,\ \mathcal{I}(w_0 + E)\rangle|^2 \\ &- \omega_k^{-1}\, \mathrm{Re}\,\langle E_k,\ \mathcal{I}(w_0 + E)\rangle\big]. \end{aligned}$$

(2) *For* $k = 1, 2, \dots, r$ *one has*

$$\begin{aligned} (H_k^2 + E_k^2)\, a_\lambda(n_0 h_0) = \lambda a_\lambda(n_0 h_0)\big[&\,4\lambda\, |\langle E_k,\ \mathcal{I}(w_0 + E)\rangle|^2 + c_k^2 \lambda \\ &- 2(2c_k\lambda + 1) \cdot \mathrm{Re}\,\langle E_k,\ \mathcal{I}(w_0 + E)\rangle\big]. \end{aligned}$$

Proof. (1) Since $\langle JT, E_{\mathbf{c}}^* \rangle = 0$ and since $[T, JT] = -\omega_k^{-1} E_k$, the formula is a direct consequence of Lemmas 4.4 and 4.5.

(2) This results also from Lemmas 4.4 and 4.5, because $[H_k, E_k] = E_k$. □

Lemma 4.7. *For $Z \in \mathfrak{g}(1/2)$, one has*

$$\begin{aligned}Z^2 a_\lambda(n_0h_0) = \lambda a_\lambda(n_0h_0)\big[4\lambda\big(&\operatorname{Re}\langle Q(u_0, Z),\ \mathfrak{I}(w_0+E)\rangle\big)^2\\ &- 2\operatorname{Re}\langle Q(Z,Z),\ \mathfrak{I}(w_0+E)\rangle\\ &- 2\operatorname{Re}\langle Q(u_0,Z),\ (D_{Q(u_0,Z)}\mathfrak{I})(w_0+E)\rangle\big].\end{aligned}$$

Proof. Since $\exp tZ = n(0, tZ)$, we have by (2.11)

$$\exp\ (-tZ)\, n_0h_0 \cdot \mathrm{e} = (u_0 - tZ,\ w_0 - tQ(u_0, Z) + \tfrac{1}{2}t^2 Q(Z, Z)). \tag{4.9}$$

Hence it holds by virtue of Lemma 4.3 that

$$\begin{aligned}Za_\lambda(n_0h_0) &= 4^{|\mathbf{c}|\lambda}\chi_{\mathbf{c}}(h_0)^\lambda \frac{d}{dt}\big[\eta(w_0 + E - tQ(u_0, Z))^\lambda\\ &\qquad\cdot \eta(w_0^* + E - tQ(Z, u_0))^\lambda\big]\Big|_{t=0}\\ &= 2\lambda a_\lambda(n_0h_0)\cdot \operatorname{Re}\langle Q(u_0, Z),\ \mathfrak{I}(w_0+E)\rangle.\end{aligned}$$

In view of (4.9) this leads us to

$$\begin{aligned}Z^2a_\lambda(n_0h_0) &= 2\lambda\cdot Za_\lambda(n_0h_0)\cdot \operatorname{Re}\langle Q(u_0, Z),\ \mathfrak{I}(w_0+E)\rangle\\ &\quad + 2\lambda a_\lambda(n_0h_0)\cdot \operatorname{Re}\frac{d}{dt}\big\langle Q(u_0 - tZ, Z),\ \mathfrak{I}(w_0 - tQ(u_0,Z) + E)\big\rangle\Big|_{t=0},\end{aligned}$$

from which the lemma follows without difficulty. □

Proposition 4.8. *If $Z \in \mathfrak{n}_{\alpha_k/2}$ with $\|Z\|_\omega = 1$, it holds that*

$$\begin{aligned}(Z^2 + (JZ)^2)\, a_\lambda(n_0h_0) = 2\lambda a_\lambda(n_0h_0)\big[&2\lambda\,|\langle Q(u_0, Z),\ \mathfrak{I}(w_0+E)\rangle|^2\\ &- \omega_k^{-1}\operatorname{Re}\langle E_k,\ \mathfrak{I}(w_0+E)\rangle\big].\end{aligned}$$

Proof. Recall that Q is sesqui-linear on $U \times U$ by (2.8). Moreover we have $Q(Z, Z) = \frac{1}{2}[JZ, Z] = \frac{1}{2}\omega_k^{-1}E_k$. The corollary then follows from Lemma 4.7 and these observations. □

Let $\psi : G \to D$ be the surjective diffeomorphism given by the orbit map $\psi(g) := g\cdot \mathrm{e}$. Using (2.6) and (2.11), we see that its differential $d\psi : \mathfrak{g} \to U + W$ at the identity is described as

$$d\psi(T + u + x) = u + (-JT + ix),\qquad T \in \mathfrak{g}(0),\ u \in \mathfrak{g}(1/2),\ x \in \mathfrak{g}(1).$$

Let us regard $\mathfrak{g}$ as a complex vector space by means of $-J$. Then, as is easily verified, $d\psi$ is complex linear, that is, $d\psi(-JX) = i\cdot d\psi(X)$ for all $X \in \mathfrak{g}$. We equip the complex vector space $(\mathfrak{g}, -J)$ with a Hermitian inner product $(\cdot\,|\,\cdot)_\omega$ defined by

$$(X\,|\,Y)_\omega := \langle [JX, Y], \omega\rangle - i\langle [X, Y], \omega\rangle.$$

We then transport it to $U + W$ by $d\psi$ and get a Hermitian inner product $(\cdot \mid \cdot)_\omega$ on $U + W$. Denoting by the same symbol the complex bilinear extension to $W \times W$ of the real inner product $\langle \cdot \mid \cdot \rangle_\omega$ of V, we have by an easy computation

$$(X + iY \mid X' + iY')_\omega = (JX + Y \mid JX' + Y')_\omega = \langle X + iY \mid X' - iY' \rangle_\omega.$$

where $X, X', Y, Y' \in V$. Identifying $U^\dagger + W^*$ with $U + W$ through $(\cdot \mid \cdot)_\omega$, we get an inner product on $U^\dagger + W^*$, which we still denote by $(\cdot \mid \cdot)_\omega$.

Let $\mathcal{C}$ be the Cayley transform defined by (2.23).

Proposition 4.9. *One has*

$$(\Lambda - \Psi)\, a_\lambda(g_0) = \lambda a_\lambda(g_0)\big[\lambda \|\mathcal{C}(g_0 \cdot \mathrm{e})\|_\omega^2 - \langle \Psi, \alpha_{\mathbf{c}} \rangle\big], \qquad g_0 \in G.$$

Proof. To calculate Λa_λ, let us choose an orthonormal basis of $\mathfrak{g}$ relative to the real inner product $\langle \cdot \mid \cdot \rangle_\omega$ of (2.2). In view of (2.7) we first put for $k = 1, \ldots, r$

$$A_k := \omega_k^{-\frac{1}{2}} H_k, \qquad F_k := \omega_k^{-\frac{1}{2}} E_k.$$

It is clear that $\{A_k\}$ forms an orthonormal basis of $\mathfrak{a}$. Let $m := \dim_{\mathbb{C}} U = \sum b_k$ and fix an orthonormal basis $Z_1, \ldots, Z_m$ of U relative to $(\cdot \mid \cdot)_\omega$. Then

$$Z_1, \ldots, Z_m, JZ_1, \ldots, JZ_m \qquad \text{form an orthonormal basis of } \mathfrak{g}(1/2).$$

Finally we choose an orthonormal basis $\{T_j\}$ of $\mathfrak{n}_0 = \mathfrak{g}(0) \cap \mathfrak{a}^\perp$ consisting of unit root vectors. Then $\{JT_j\}$ together with the above $\{F_k\}$ forms an orthonormal basis of $\mathfrak{g}(1) = V$.

Now we multiply the formula in (2) of Proposition 4.6 by ω_k^{-1} and sum up the formula over $k = 1, 2, \ldots, r$. Then we get

$$\begin{aligned}\sum_{k=1}^{r} (A_k^2 + F_k^2)\, a_\lambda(n_0 h_0) = \lambda a_\lambda(n_0 h_0)\Big[& 4\lambda \sum |\langle F_k,\, \mathfrak{I}(w_0 + E)\rangle|^2 \\ & + \lambda \sum c_k^2 \omega_k^{-1} - 2 \sum (2c_k\lambda + 1)\, \omega_k^{-1} \cdot \mathrm{Re}\,\langle E_k,\, \mathfrak{I}(w_0 + E)\rangle \Big].\end{aligned}$$

Similarly we have by (1) of Proposition 4.6 and (2.7)

$$\begin{aligned}\sum (T_j^2 + (JT_j)^2)\, a_\lambda(n_0 h_0) &= \lambda a_\lambda(n_0 h_0) \\ &\times \Big[4\lambda \sum |\langle JT_j,\, \mathfrak{I}(w_0 + E)\rangle|^2 - 2 \sum_{k=1}^{r} \omega_k^{-1} q_k \cdot \mathrm{Re}\,\langle E_k,\, \mathfrak{I}(w_0 + E)\rangle \Big].\end{aligned}$$

Finally Proposition 4.8 together with (2.7) gives

$$\begin{aligned}\sum (Z_j^2 + (JZ_j)^2)\, a_\lambda(n_0 h_0) &= \lambda a_\lambda(n_0 h_0) \\ &\times \Big[4\lambda\, \|\mathfrak{I}(w_0 + E) \circ Q(u_0, \cdot)\|_\omega^2 - 2 \sum_{k=1}^{r} \omega_k^{-1} b_k \cdot \mathrm{Re}\,\langle E_k,\, \mathfrak{I}(w_0 + E)\rangle \Big].\end{aligned}$$

Summing up these three formulas, we obtain

$$\Lambda a_\lambda(n_0h_0) = \lambda a_\lambda(n_0h_0)\Big[4\lambda\,\|\mathcal{I}(w_0+E)\|_\omega^2 + 4\lambda\,\|\mathcal{I}(w_0+E)\circ Q(u_0,\cdot)\|_\omega^2 + \lambda\sum_{k=1}^r c_k^2\omega_k^{-1} - 2\sum_{k=1}^r \omega_k^{-1}(q_k+b_k+1+2\lambda c_k)\cdot \mathrm{Re}\,\langle E_k,\ \mathcal{I}(w_0+E)\rangle\Big].$$

On the other hand, since $\|E_k^*\|_\omega^2 = \omega_k^{-1}$ for any k as is easily seen, we have by (2.22)

$$\begin{aligned}\|C(w_0)\|_\omega^2 &= \|E_{\mathbf{c}}^* - 2\,\mathcal{I}(w_0+E)\|_\omega^2\\ &= \sum_{k=1}^r c_k^2\omega_k^{-1} - 4\,\mathrm{Re}\,(\mathcal{I}(w_0+E)\mid E_{\mathbf{c}}^*)_\omega + 4\,\|\mathcal{I}(w_0+E)\|_\omega^2.\end{aligned}$$

Since $\big(\mathcal{I}(w_0+E)\mid E_{\mathbf{c}}^*\big)_\omega = \sum_{k=1}^r \omega_k^{-1}c_k\langle E_k,\ \mathcal{I}(w_0+E)\rangle$, the preceding two formulas and (2.23) lead us to

$$\begin{aligned}&\Lambda a_\lambda(n_0h_0)\\ &\quad= \lambda a_\lambda(n_0h_0)\Big[\lambda\,\|\mathcal{C}(z_0)\|_\omega^2 - 2\sum_{k=1}^r \omega_k^{-1}(q_k+b_k+1)\cdot\mathrm{Re}\,\langle E_k,\ \mathcal{I}(w_0+E)\rangle\Big]\\ &\quad= \lambda a_\lambda(n_0h_0)\big[\lambda\,\|\mathcal{C}(z_0)\|_\omega^2 + 2\,\mathrm{Re}\,\langle J\Psi,\ \mathcal{I}(w_0+E)\rangle\big],\end{aligned}$$

where we have put $z_0 := n_0h_0\cdot \mathsf{e} = (u_0, w_0)$, and the second equality follows from (see Lemma 3.3)

$$J\Psi = -\sum_{k=1}^r \omega_k^{-1}(q_k+b_k+1)\,E_k.$$

To finish the proof it suffices to note that (4.7) and (4.8) show

$$\Psi a_\lambda(n_0h_0) = \lambda a_\lambda(n_0h_0)\big[\langle\Psi,\alpha_{\mathbf{c}}\rangle + 2\,\mathrm{Re}\,\langle J\Psi,\ \mathcal{I}(w_0+E)\rangle\big].$$

Hence we get the formula in the proposition. □

Let $\beta\in\mathfrak{g}^*$ be the *Koszul form* given by

$$\langle x,\beta\rangle := \mathrm{tr}(\mathrm{ad}(Jx) - J\ \mathrm{ad}(x)),\qquad x\in\mathfrak{g}. \tag{4.10}$$

We know that β is admissible and that the inner product $\langle\cdot\mid\cdot\rangle_\beta$ is the real part of the Hermitian inner product on $\mathfrak{g}$ induced by the Bergman metric of D up to a positive multiple (see [12, Théorème 1]).

Theorem 4.10. *Let $\lambda>\lambda_0$ be fixed. Then, B_λ commutes with $\mathcal{L}_\omega$ if and only if D is symmetric and $\omega|_{\mathfrak{n}}$ is equal to a positive number multiple of $\beta|_{\mathfrak{n}}$.*

Proof. In view of Propositions 4.2 and 4.9, B_λ commutes with $\mathcal{L}_\omega$ if and only if

$$\|\mathcal{C}(g\cdot\mathsf{e})\|_\omega = \|\mathcal{C}(g^{-1}\cdot\mathsf{e})\|_\omega\qquad\text{for any } g\in G,$$

where recall that $a_\lambda = a_\lambda^\vee$. Therefore the theorem is a consequence of the following theorem established in [15]. $\square$

Theorem 4.11 ([15]). *One has $\|\mathcal{C}(g \cdot \mathsf{e})\|_\omega = \|\mathcal{C}(g^{-1} \cdot \mathsf{e})\|_\omega$ for all $g \in G$ if and only if D is symmetric and $\omega|_{\mathfrak{n}}$ is equal to a positive number multiple of $\beta|_{\mathfrak{n}}$.*

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